Mathematics Online Summer Academy (MOSA)

Experience Gauger

Application Deadline: June 15, 2021

Hello dearest MOSA applicants:

Welcome to MOSA! Our mission in creating MOSA is to provide middle schoolers who are interested in exploring mathematics topics an opportunity to learn the more advanced mathematics outside of a classroom. We welcome applicants from all backgrounds, regardless of gender, race, or ethnicity, and we strongly encourage minorities, girls, and those with little to no mathematical experience to apply.

The **Experience Gauger** is a problem set consisting of 7 problems covering a variety of mathematical topics. You might find some of the problems very simple and straightforward, while others slightly harder. If you find the problems too simple, we highly recommend that you look into other mathematics summer programs which require prior mathematical experience and have more challenging admissions exams.

The problem set is meant to give us a better indication as to how we should design our curriculum and to kindly remind advanced students with more mathematical experience that this program might not be the best way for you to spend your summer. If this problem set seems challenging, then MOSA is the right program for you!

Please work on the problems **independently** for as long as you want. You may consult the internet or books for unfamiliar terms and topics as long as you reference them or link the source, and you **cannot** ask other human beings for help. If you have seen a certain problem before, please indicate so when applying. For each problem, feel free to explore different approaches such as drawing diagrams and tables, making guesses, or testing out cases. Go crazy with experimenting! We are not looking for fully written-out solutions and we do not require that you answer all of the problems, just that you try. Include your thought processes and any progress you have made in your write-up (but not your scratch paper), even if you did not completely solve the problem. Since admissions is on a rolling basis, please plan your time wisely.

For Experience Gauger submission, you can either LaTeX your solution, or handwrite it, take pictures, and convert the output or file into a PDF. **We only accept PDF files!** If you are handwriting the solution, please make sure that it is legible. If your submission does not follow the guidelines above, we might contact you for a second submission. You can also contact us at MOSAadmissions@gmail.com or message us at +1 646-464-5336 if you have any questions. If you are messaging the phone number, please indicate that you are a MOSA applicant first.

Last but not least, **enjoy the math!**

Just remember, the number of problems you solved on the E.G. will not directly affect your chances of being admitted to MOSA. In fact, MOSA is intended to cover those topics on the E.G. We only encourage you to try your best.

Problems

Problem 1 Calculate the last digit of the following integer.

$$314159265358^{2021}$$

Problem 2 M, A, T, H are four positive integers in the following vertical addition.

$$\begin{array}{rcccc} & \mathrm{M} & \mathrm{A} & \mathrm{T} & \mathrm{H} \\ & & \mathrm{A} & \mathrm{T} & \mathrm{H} \\ & & & \mathrm{T} & \mathrm{H} \\ + & & & & \mathrm{H} \\ \hline & \mathrm{T} & \mathrm{H} & \mathrm{M} & \mathrm{A} \end{array}$$

Calculate the four-digit integer $\overline{\mathrm{MATH}}$.

Problem 3 In the 2-dimensional world of Corona Village, where every citizen is a coronavirus (let them be perfect circles) with a radius of 1 foot. In a square classroom with side length 40 feet, there are already four students sitting at each corner. If a fifth student walks in and everyone practices proper social distancing by staying at least 6 feet apart, the new student can only travel within a fixed boundary to stay safe. What is the area of this "safe" zone?

Problem 4 Let $S(N)$ denote to the sum of the digits of the integer N. Now, there exists a five-digit number N such that $\sqrt{N} = 9 \cdot S(N)$. Find all N.

Problem 5 Ted is making flower bouquets for his pet pony's birthday and needs to purchase roses and magnolias. The company that sells Ted flowers only delivers roses in packs of 5 flowers and magnolias in packs of 7 flowers. What is the largest number of flowers Ted cannot purchase? For example, Bob can order 12 flowers (5 roses + 7 magnolias) but cannot purchase 16 flowers.

(a) Start out with some smaller cases. What are some numbers Bob can definitely purchase?

(b) The diagram below lists all numbers that Ted could or could not purchase. Start crossing out numbers Ted can purchase. What do you notice?

0	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31	32	33	34
35	36	37	38	39	40	41

(c) How can we generalize your observation to roses in batches of n and magnolias in batches of m?

Problem 6

(a) Calculate the following sums. What have you found?

$$\begin{aligned} S_1 &= 1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 \\ S_2 &= 1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + 9 \times 10. \end{aligned}$$

We now introduce a method to solve sums like S_1, S_2.

Telescoping. To solve the sum

$$S = 1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + n \times (n+1),$$

we consider each term $k(k+1)$. We can write $k(k+1)$ as

$$\begin{aligned} k(k+1) &= \frac{(k+2)k(k+1) - (k-1)k(k+1)}{(k+2)-(k-1)} \\ &= \frac{1}{3}\left(k(k+1)(k+2) - (k-1)k(k+1)\right). \end{aligned}$$

Therefore,

$$\begin{aligned} S &= \frac{1}{3}((1 \times 2 \times 3 - 0 \times 1 \times 2) + (2 \times 3 \times 4 - 1 \times 2 \times 3) \\ &\quad + \cdots + (n(n+1)(n+2) - (n-1)n(n+1)) \\ &= \frac{n(n+1)(n+2) - 0 \times 1 \times 2}{3} \\ &= \frac{n(n+1)(n+2)}{3}. \end{aligned}$$

Telescoping has a variety of applications in all kinds of math problems, one of which is to deduce the formula for the sum of consecutive squares.

(b) Use Telescoping to deduce the formula for the sum of consecutive numbers.

$$S = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Now lets apply this formula to Problem 6(c):

(c) Nancy writes the positive integers from 1 through 2021 on a whiteboard. A *move* consists of erasing two numbers a, b and writing $\sqrt{a^2 + b^2}$ on the whiteboard. Angela makes moves until the whiteboard has only one number left. Find the maximum possible value of the last number. (*Hint:* Try smaller cases. What do you notice if Nancy writes the positive integers from 1 through 3? How can we apply these observations to the original problem?)

Problem 7

(a) Anoushka wants to fill a 1×10 grid with tiles, but she has only 1×1 tiles and 1×2 tiles. How many ways are there to fill the grids without overlap?

(b) Natalia wants to fill a 2×10 grid with tiles, but she has only 1×1 tiles and 2×2 tiles with a 1×1 corner missing. How many ways are there to fill the grids without overlap?

(c) Natalia and Anoushka now want to work together to fill a 3×10 grid with tiles, but they have only 1×1 tiles and 2×3 tiles with a 2×1 corner missing (so look like L). How many ways are there to fill the grids without overlap?

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